Proposition. *Let* (a_n) *be a sequence of real numbers,* $a \in \mathbb{R}$*. Then* $\lim_{n\to\infty} a_n = a$ *if and only if for any* $\varepsilon > 0$ *the set*

$$B_\varepsilon := \{n \,:\, a_n \notin V_\varepsilon(a)\}$$

is finite.

Proof. Assume first that $\lim_{n\to\infty} a_n = a$. Fix $\varepsilon > 0$. Then for some $N \in \mathbb{N}$, $a_n \in V_\varepsilon(a)$ if $n \geq N$. Thus any such n do not belong to B_ε, so B_ε is a subset of a finite set $\{1, 2, \ldots N-1\}$. So B_ε itself is finite.

Let us now assume that B_ε is finite for any $\varepsilon > 0$. Again, fix $\varepsilon > 0$. Take $N = \max B_\varepsilon + 1$. Then if $n \geq N$, $n \notin B_\varepsilon$, so $a_n \in V_\varepsilon$. Thus

$$\forall \varepsilon > 0 \, \exists N \,:\, n \geq N \implies a_n \in V_\varepsilon(a).$$

So $\lim_{n\to\infty} a_n = a$. □